THREE-DIMENSIONAL NTH DERIVATIVE OF GAUSSIAN SEPARABLE STEERABLE FILTERS

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ABSTRACT

This paper details the construction of three-dimensional separable steerable filters. The approach presented is an extension of the construction of two-dimensional separable steerable filters outlined in [1]. Additionally, three-dimensional separable steerable filters, both continuous and discrete versions, for the second derivative of the Gaussian and its Hilbert transform are reported. Experimental evaluation demonstrates that the errors in the constructed separable filters are negligible.

1 INTRODUCTION

In many vision and image processing tasks the application of filters at arbitrary orientations is made. For example in [2], the authors show that motion manifests itself as orientation in the spatiotemporal domain and use spatiotemporally oriented filters to detect it. A computationally expensive approach is the application of many rotated versions of a filter differing by a small rotation angle. In [1], the authors demonstrate that for a certain class of functions (i.e., filters), rotated copies can be synthesized by taking linear combinations of a small set of basis functions (i.e., the basis functions span the space of all rotations). Furthermore, the authors leverage the distributive property of linear filters by first convolving the input image with the set of basis functions and then realizing the required filtered version of the image by taking appropriate linear combinations of the outputs. Additionally, the authors present the construction of two-dimensional separable filters (for polynomial functions) that provide a significant gain in computational efficiency over their non-separable equivalents. Further, related work to the steerable filters considered in this paper include, early work in image restoration (e.g., [3]), analyzing oriented patterns (e.g., [4]), alternative analytic descriptions of steerable filters (e.g., [5, 6, 7, 8]), generating steerable filters through SVD approximations (e.g., [9, 10, 11]) and work considering various orders of Gaussian derivatives (e.g., [12, 13, 14]).

The current paper provides explicit forms for three dimensional (e.g., X-Y-Z) nth degree separable steerable filters. The approach presented is an extension of the construction of two-dimensional (X-Y) separable steerable filters for polynomial functions outlined in [1]. As examples, both analytic and discrete forms for the second derivative of a Gaussian (G_2) and its Hilbert transform (H_2) are provided. Additionally, numerical evaluations of the discrete separable versions of the G_2 and H_2 filters are provided. It appears that explicit analytic and numerical formulations for these filters have not previously appeared.

In this paper it is assumed that the functions to be steered are of the form of a polynomial times a separable windowing function. Additionally, the functions are assumed to have an axis of rotational symmetry. These functions, rotated by a transformation $\mathbf{R}$ such that their axis of symmetry point along the direction cosines α, β and γ, can be written as,

$$f^{\mathbf{R}}(x,y,z) = W(r)P_N(x') \tag{1}$$

where $W(r)$ is any spherically symmetric function (e.g., a three-dimensional Gaussian-like function: e^{-r^2}, $r = \sqrt{x^2+y^2+z^2}$) and $P_N(x')$ is an nth order polynomial in

$$x' = \alpha x + \beta y + \gamma z \tag{2}$$

The following theorem (Theorem 4 in [1]) provides the means for constructing steerable filters of axially symmetric three-dimensional functions.

Theorem 1 *Given a three dimensional axially symmetric function $f(x,y,z) = W(r)P_N(x)$, where $P_N(x)$ is an even or odd symmetry nth order polynomial in x. Let α, β and γ be the direction cosines of the axis of symmetry of $f^{\mathbf{R}}(x,y,z)$ and α_j, β_j and γ_j be the direction cosines of the axis of symmetry of $f^{\mathbf{R}_j}(x,y,z)$. Then the steering equation,*

$$f^R(x,y,z) = \sum_{j=1}^{M} k_j(\alpha,\beta,\gamma) f^{\mathbf{R}_j}(x,y,z), \tag{3}$$

holds if and only if

1. *$M \geq (N+1)(N+2)/2$ and*
2. *the $k_j(\alpha,\beta,\gamma)$ satisfy*

$$\begin{pmatrix} \alpha^N \\ \alpha^{N-1}\beta \\ \alpha^{N-1}\gamma \\ \alpha^{N-2}\beta^2 \\ \\ \vdots \\ \\ \gamma^N \end{pmatrix} = \begin{pmatrix} \alpha_1^N & \alpha_2^N & \dots & \alpha_M^N \\ \alpha_1^{N-1}\beta_1 & \alpha_2^{N-1}\beta_2 & \dots & \alpha_M^{N-1}\beta_M \\ \alpha_1^{N-1}\gamma_1 & \alpha_2^{N-1}\gamma_2 & \dots & \alpha_M^{N-1}\gamma_M \\ \alpha_1^{N-2}\beta_1^2 & \alpha_2^{N-2}\beta_2^2 & \dots & \alpha_M^{N-2}\beta_M^2 \\ \\ \vdots & \vdots & \vdots & \vdots \\ \\ \gamma_1^N & \gamma_2^N & \dots & \gamma_M^N \end{pmatrix} \begin{pmatrix} k_1(\alpha,\beta,\gamma) \\ k_2(\alpha,\beta,\gamma) \\ k_3(\alpha,\beta,\gamma) \\ \dots \\ k_M(\alpha,\beta,\gamma) \end{pmatrix} \tag{4}$$

2 BASIS FUNCTIONS SEPARABLE IN X, Y and Z

We consider the case of even or odd filters $f^{\Omega}(x, y, z)$ which can be written as,

$$f^{\Omega}(x, y, z) = G(r)Q_N(x') \tag{5}$$

where $G(r)$ is a separable windowing function (e.g., Gaussian-like function $e^{-r^2} = e^{-(x^2+y^2+z^2)} = e^{-x^2}e^{-y^2}e^{-z^2}$) and $Q_N(x')$ is an nth order polynomial in,

$$x' = \alpha x + \beta y + \gamma z \tag{6}$$

(c.f., [1] that does not deal explicitly with 3D separability, only 2D).

By Theorem 1, $(N + 1)(N + 2)/2$ functions can form a basis set for $f^{\Omega}(x, y, z)$. We assume that a basis of $M = (N + 1)(N + 2)/2$ X-Y-Z separable functions exists. Then there will be some set of separable basis functions $Q_{i,j}(x)R_{i,j}(y)S_{i,j}(z)$ for which

$$f^{\Omega}(x, y, z) = G(r)\sum k_{i,j}(\Omega)Q_{i,j}(x)R_{i,j}(y)S_{i,j}(z) \tag{7}$$

The interpolation functions, $k_{i,j}(\Omega)$ are found by equating the highest order products of x, y and z in Eq. (5) with those in Eq. (7), i.e., equating the coefficients of $x^{N-i-j}y^iz^j$ for $\{i, j|i, j \in \mathbb{N} \text{ and } i + j \leq N\}$. Substituting Eq. (6) into Eq. (5), yields M different products of x, y and z of order N, since (by the specialization of the multinomial theorem [15]),

$$(x')^N = \sum \frac{n!}{(n-i-j)!i!j!}\alpha^{N-i-j}\beta^i\gamma^j x^{N-i-j}y^iz^j \tag{8}$$

where the summation is taken over terms with all possible integer values of i, j between 0 and N subject to the constraint that $i + j \leq N$.

Each basis function $Q_{i,j}(x)R_{i,j}(y)S_{i,j}(z)$ can contribute only one product of powers of x, y and z of order N (otherwise $Q_{i,j}(x)R_{i,j}(y)S_{i,j}(z)$ would be a polynomial of x, y and z of order higher than N). So we have

$$Q_{i,j}(x)R_{i,j}(y)S_{i,j}(z) = c(x^{N-i-j}+\cdots)(y^i+\cdots)(z^j+\cdots), \tag{9}$$

where c is a constant. Therefore, Eq. (5) shows that the coefficients of the highest order terms $x^{N-i-j}y^iz^j$, in $f^{\Omega}(x, y, z)$ are

$$k_{i,j}(\alpha, \beta, \gamma) = \frac{n!}{(n-i-j)!i!j!}\alpha^{N-i-j}\beta^i\gamma^j. \tag{10}$$

Note that the lower order terms can appear in more than one separable basis function, so their coefficients will be the result of a sum of different $k_{i,j}(\alpha, \beta, \gamma)$.

To find the separable basis functions $Q_{i,j}(x)R_{i,j}(y)S_{i,j}(z)$ from the original function $f(x, y, z)$, we note that from the steering equation for the separable basis functions, Eq. (7), we have,

$$\begin{pmatrix} f^{\Omega_1}(x,y,z) \\ f^{\Omega_2}(x,y,z) \\ \vdots \\ f^{\Omega_M}(x,y,z) \end{pmatrix} = G(r) \begin{pmatrix} \cdots & k_{i,j}(\Omega_1) & \cdots \\ \cdots & k_{i,j}(\Omega_2) & \cdots \\ \vdots & \vdots & \vdots \\ \cdots & k_{i,j}(\Omega_M) & \cdots \end{pmatrix} \begin{pmatrix} \vdots \\ Q_{i,j}(x)R_{i,j}(y)S_{i,j}(z) \\ \vdots \end{pmatrix} \tag{11}$$

The $Q_{i,j}(x)R_{i,j}(y)S_{i,j}(z)$ can be written as a linear combination of f^{Ω_j} by inverting the matrix of $k_{i,j}$'s of Eq. (11).

3 G_2/H_2 STEERABLE QUADRATURE FILTER PAIRS

A three-dimensional Gaussian-like function can be written as,

$$G(x, y, z) = e^{-(x^2+y^2+z^2)}. \tag{12}$$

The second derivative with respect to x of the three-dimensional Gaussian-like function is written as,

$$\frac{\partial^2 G}{\partial x^2} = (4x^2 - 2)e^{-(x^2+y^2+z^2)}. \tag{13}$$

Using Theorem 1 where $N = 2$, we need 6 basis functions. Tables 1, 2 and 3 were arrived at by using the construction outlined in Section 2. The direction cosines α, β and γ used were a distinct subset of the unit normals to the faces of the dodecahedron (or the vertices of the icosahedron), which can be generated according to a cyclic permutation of $(\pm 1, 0, \pm r)$ where r is the golden mean, $(\sqrt{5} + 1)/2$ [16]. The dodecahedron provides a uniform 12-face tessellation of a sphere.

The Hilbert transform[1] of the second derivative of the three-dimensional Gaussian function is written as,

$$H_2(x, y, z) = (-2.254x + x^3)e^{-(x^2+y^2+z^2)}. \tag{14}$$

Using Theorem 1 where $N = 3$, we need 10 basis functions. Tables 4, 5 and 6 were arrived at by using the construction outlined in Section 2. The direction cosines α, β and γ used were a distinct subset of the unit normals to the faces of the icosahedron (or the vertices of the dodecahedron), which can be generated according to a cyclic permutation of $(0, \pm r, \pm 1/r)$ and $(\pm 1, \pm 1, \pm 1)$ where r is the golden mean [16].

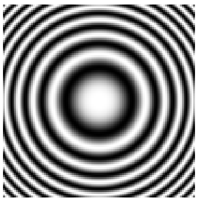


Figure 1: Zone plate. Depicted is a X-Y slice from a three-dimensional zone plate.

4 EXPERIMENTAL EVALUATION

For each of the filters we compared the convolution output of a test image with the rotated versions of the non-separable and separable filters. The test image used was a three-dimensional zone plate, specifically a 3D analog of the 1D linear chirp (i.e., $f(x) = cos[(\omega x)x] = cos(\omega x^2)$, where ωx denotes the instantaneous frequency), defined as,

$$\begin{aligned} f(x, y, z) &= cos[(\omega\sqrt{x^2 + y^2 + z^2})\sqrt{x^2 + y^2 + z^2}] \\ &= cos[\omega(x^2 + y^2 + z^2)] \end{aligned} \quad (15)$$

Given the orientation and frequency selectivity of the filters, the zone plate was selected because it captures a continuum in both orientations and frequencies above and beyond those from which the individual filters are selective. In Fig. 1 a two-dimensional slice of the three-dimensional zone plate is presented. The rotated versions of the non-separable filters were arrived at by conducting the rotations in the continuous domain followed by discretization.

The sampling of the orientation of the filters were arrived at by representing the direction cosines (α, β, γ) of the axis of symmetry parametrically by spherical coordinates (θ, ϕ) as follows,

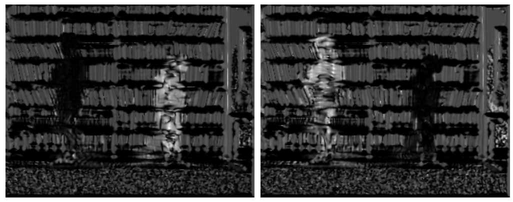
$$\alpha = cos(\theta)sin(\phi),\ \beta = sin(\theta)sin(\phi),\ \gamma = cos(\phi) \quad (16)$$

and sampling θ and ϕ at $5°$ intervals with the following bounds, $0 \leq \theta \leq 2\pi$ and $-\pi/2 \leq \phi \leq \pi/2$. The maximum rms errors for the G_2 and H_2 were found to be 6.63×10^{-13} and 3.55×10^{-12} across all orientations, respectively.

To provide one indication of the practical applicability of the desired filters, video surveillance is considered. Early delineation of coherently moving structures can provide a strong cue to target detection and tracking [17]. Correspondingly, Fig. 2 illustrates the delineation of coherently moving structures in a video sequence through the application of the 3D steerable Gaussian filters and the their Hilbert transforms applied in quadrature to extract energy indicative of righward and lefward moving targets, see [18] for details.

[1]The Hilbert transform of the second derivative of the three-dimensional Gaussian function was approximated by finding the least squares fit to a third-order polynomial times a Gaussian [1].

(a) intensity image



(b) rightward (c) leftward

Figure 2: Coherence images from an indoor scene. (a) frame taken from a scene of two people walking away from each other, one rightward and the other leftward. Highlighted in white (b) the rightward coherent structures, (c) the leftward coherent structures.

5 SUMMARY

In this paper the construction of three-dimensional separable steerable filters of the nth derivative of the Gaussian was presented. Additionally, as a proof of concept the second derivative of the Gaussian and its Hilbert transform, both in the continuous and discrete domains, were presented. Experimental evaluation shows that the error in the construction of the separable steerable second derivative of the Gaussian and Hilbert transform is negligible.

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Basis	Interpolation
$G_{2a} = N(2x^2 - 1)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = \alpha^2$
$G_{2b} = N(2xy)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = 2\alpha\beta$
$G_{2c} = N(2y^2 - 1)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = \beta^2$
$G_{2d} = N(2xz)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = 2\alpha\gamma$
$G_{2e} = N(2yz)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = 2\beta\gamma$
$G_{2f} = N(2z^2 - 1)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = \gamma^2$

Table 1: X-Y-Z separable basis set and interpolation functions for the second derivative of the Gaussian. N is a normalization constant equaling $\frac{2}{\sqrt{3}}(\frac{2}{\pi})^{3/4}$, introduced so that the integral over all space of the square of the function equals one. To construct a second derivative of a Gaussian where the axis of symmetry (i.e., X-axis) is mapped to the direction cosine $\Omega = (\alpha, \beta, \gamma)$, use $G_2^\Omega = \sum_{i \in \{a,\ldots,f\}} k_i(\Omega) G_{2i}$.

	1D Function	Tap #				
		0	1	2	3	4
f1	$N(2t^2 - 1)e^{-t^2}$	-0.8230	-0.0537	0.3540	0.1025	0.0084
f2	e^{-t^2}	1.0000	0.6383	0.1660	0.0176	0.0008
f3	$2Nte^{-t^2}$	0	0.7039	0.3662	0.0582	0.0034
f4	te^{-t^2}	0	0.4277	0.2225	0.0354	0.0020

Table 2: 9-tap filters for X-Y-Z separable basis set for G_2. Filters f1 and f2 have even symmetry; f3 and f4 have odd symmetry. These filters were taken from Table 1, with a sample spacing of 0.67.

G_2 Basis Filter	Filter in X	Filter in Y	Filter in Z
G_{2a}	f1	f2	f2
G_{2b}	f3	f4	f2
G_{2c}	f2	f1	f2
G_{2d}	f3	f2	f4
G_{2e}	f2	f3	f4
G_{2f}	f2	f2	f1

Table 3: G_2 basis filters. Summarized is the construction of the G_2 basis filters (a-f) using the filters given in Table 2.

Basis	Interpolation
$H_{2a} = N(x^3 - 2.254x)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = \alpha^3$
$H_{2b} = Ny(x^2 - 0.751333)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = 3\alpha^2\beta$
$H_{2c} = Nx(y^2 - 0.751333)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = 3\alpha\beta^2$
$H_{2d} = N(y^3 - 2.254y)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = \beta^3$
$H_{2e} = Nz(x^2 - 0.751333)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = 3\alpha^2\gamma$
$H_{2f} = Nxyze^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = 6\alpha\beta\gamma$
$H_{2g} = Nz(y^2 - 0.751333)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = 3\beta^2\gamma$
$H_{2h} = Nx(z^2 - 0.751333)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = 3\alpha\gamma^2$
$H_{2i} = Ny(z^2 - 0.751333)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = 3\beta\gamma^2$
$H_{2j} = N(z^3 - 2.254z)e^{-(x^2+y^2+z^2)}$	$k(\alpha,\beta,\gamma) = \gamma^3$

Table 4: X-Y-Z separable basis set and interpolation functions for fit to Hilbert transform of the second derivative of the Gaussian. N is a normalization constant equaling 0.877776, introduced so that the integral over all space of the square of the function equals one. To construct a second derivative of a Gaussian Hilbert transform where the axis of symmetry (i.e., X-axis) is mapped to the direction cosine $\Omega = (\alpha, \beta, \gamma)$, use $H_2^\Omega = \sum_{i \in \{a,\ldots,j\}} k_i(\Omega) H_{2i}$. Due to space limitations the filters have been truncated. For the full precision filters used in our experiments see [18].

	1D Function	Tap #				
		0	1	2	3	4
f1	$N(t^3 - 2.254t)e^{-t^2}$	0	-0.6776	-0.0895	0.0554	0.0088
f2	$N(t^2 - 0.751333)e^{-t^2}$	-0.6595	-0.1695	0.1522	0.0508	0.0043
f3	e^{-t^2}	1.0000	0.6383	0.1660	0.0176	0.0008
f4	Nte^{-t^2}	0	0.3754	0.1953	0.0310	0.0018
f5	te^{-t^2}	0	0.4277	0.2225	0.0354	0.0020

Table 5: 9-tap filters for X-Y-Z separable basis set for H_2. Filters f2 and f3 have even symmetry; f1, f4 and f5 have odd symmetry. These filters were taken from Table 4, with a sample spacing of 0.67.

H_2 Basis Filter	Filter in X	Filter in Y	Filter in Z
H_{2a}	f1	f3	f3
H_{2b}	f2	f5	f3
H_{2c}	f5	f2	f3
H_{2d}	f3	f1	f3
H_{2e}	f2	f3	f5
H_{2f}	f4	f5	f5
H_{2g}	f3	f2	f5
H_{2h}	f5	f3	f2
H_{2i}	f3	f5	f2
H_{2j}	f3	f3	f1

Table 6: H_2 basis filters. Summarized is the construction of the H_2 basis filters (a-j) using the filters given in Table 5.